MATH 312
Lecture 4: Antithesis

UNDERSTANDING INVERSES

(section 2.5 of text, we will come back to section 2.4)

Last time:

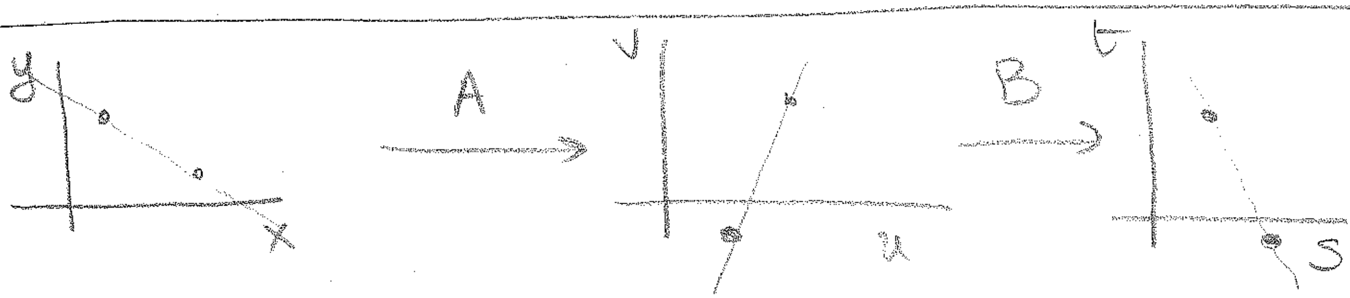
Linear System		Matrix Equation
$ax + by = u$ $cx + dy = v$	$\Longleftrightarrow$	$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

When we have another linear system (or matrix equation), say $B\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix}$, then we can relate the original input $\begin{bmatrix} x \\ y \end{bmatrix}$ to final output $\begin{bmatrix} s \\ t \end{bmatrix}$ as follows:

$$\begin{bmatrix} s \\ t \end{bmatrix} = B\begin{bmatrix} u \\ v \end{bmatrix} = \underbrace{BA}_{\text{matrix product}}\begin{bmatrix} x \\ y \end{bmatrix}$$

Now, $B = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, so

$$BA = \begin{bmatrix} a'a + b'c & a'b + b'd \\ c'a + d'c & c'b + d'd \end{bmatrix}.$$



SOME matrices are invertible, i.e., there is a matrix A^{-1} so that $A^{-1}A = Id = AA^{-1}$

identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (acts like "1")

If the matrix of a linear system is INVERTIBLE, then we can solve it no matter what the right side is:

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

So, $$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} u \\ v \end{bmatrix}.$$

And no matter what $\begin{bmatrix} u \\ v \end{bmatrix}$ is we can solve for $\begin{bmatrix} x \\ y \end{bmatrix}$ with an inverse in hand.

WARNING

NOT ALL MATRICES ARE INVERTIBLE

eg: zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

MISSION:

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Given a matrix,

- Find out if it has an inverse
- Compute that inverse if it exists

(We have ONE trick: Gaussian Elim.) from lecture 3!

eg: $A = \begin{bmatrix} 3 & 2 \\ 6 & -5 \end{bmatrix}$ (Find A^{-1})

Which a, b, c, d satisfy

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 6 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ?$$

i.e., $$\begin{bmatrix} 3a+6b & 2a-5b \\ 3c+6d & 2c-5d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

BUT This is just two linear systems:

$$\boxed{\begin{aligned} 3a + 6b &= 1 \\ 2a - 5b &= 0 \end{aligned}} \quad \text{and} \quad \boxed{\begin{aligned} 3c + 6d &= 0 \\ 2c - 5d &= 1 \end{aligned}}$$

Also, the matrix for both is $A = \begin{bmatrix} 3 & 6 \\ 2 & -5 \end{bmatrix}$.

So, we could do TWO eliminations

$$\left[\begin{array}{cc|c} 3 & 6 & 1 \\ 2 & -5 & 0 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cc|c} 3 & 6 & 0 \\ 2 & -5 & 1 \end{array}\right]$$

JUST COMBINE:

$$\left[\begin{array}{cc|cc} 3 & 6 & 1 & 0 \\ 2 & -5 & 0 & 1 \end{array}\right] = [A \mid Id]$$

This is called "Gauss-Jordan" Elimination:

$$\left[\begin{array}{cc|cc} 3 & 6 & 1 & 0 \\ 2 & -5 & 0 & 1 \end{array}\right] \xrightarrow[\text{and scaling)}]{\text{(row operations}} \left[\begin{array}{cc|cc} 1 & 0 & 5/27 & 2/9 \\ 0 & 1 & 2/27 & -1/9 \end{array}\right].$$

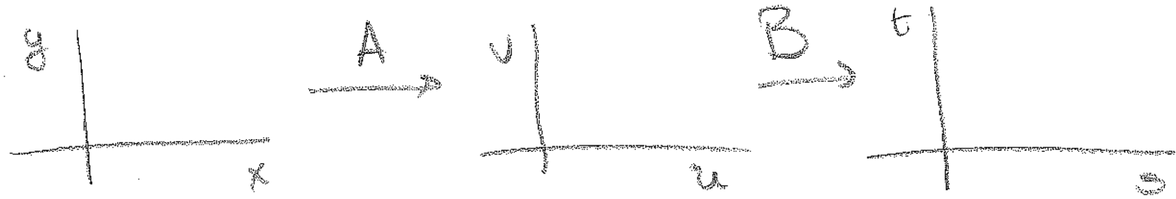
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad Id \qquad\qquad A^{-1}$

The HEART of the matter

SINCE everything is LINEAR, solving $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ gets you solve $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$! The solution for general $\begin{bmatrix} u \\ v \end{bmatrix}$ is JUST

$$u\left(\text{solution of } \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + v\left(\text{solution of } \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

When A and B are both invertible, i.e., if



Then $(BA)^{-1} = \underbrace{A^{-1} \cdot \underbrace{B^{-1}}_{\text{first undo } B}}_{\text{then undo } A}$ (ORDER is flipped!)

<u>IMPORTANT</u> An $n \times n$ matrix is INVERTIBLE if (and ONLY IF) it is "generic", i.e., the Gaussian Elimination algorithm finds n nonzero pivots.

What does matrix multiplication do to the zero vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$? Well, for any $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so.

1. No matrix, when multiplying $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, produces any nonzero vector.
2. Therefore, if $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ when $\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then there can be no A^{-1}; Any such A^{-1} has to send $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} x \\ y \end{bmatrix}$.
3. Therefore, IF some matrix A sends a NON-ZERO vector to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then it is NOT invertible. eg: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 5 & 0 \\ 6 & 0 \end{bmatrix}$, ...?